

# ON THE CONVERGENCE OF CESÀRO MEANS OF NEGATIVE ORDER OF DOUBLE TRIGONOMETRIC FOURIER SERIES OF FUNCTIONS OF BOUNDED PARTIAL GENERALIZED VARIATION

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**ABSTRACT.** The convergence of Cesàro means of negative order of double trigonometric Fourier series of functions of bounded partial  $\Lambda$ -variation is investigated. The sufficient and necessary conditions on the sequence  $\Lambda = \{\lambda_n\}$  are found for the convergence of Cesàro means of Fourier series of functions of bounded partial  $\Lambda$ -variation.

## 1. CLASSES OF FUNCTIONS OF BOUNDED GENERALIZED VARIATION

In 1881 Jordan [9] introduced the class of functions of bounded variation and applied it to the theory of Fourier series. Hereinafter this notion was generalized by many authors (quadratic variation,  $\Phi$ -variation,  $\Lambda$ -variation etc., see [2], [10], [13], [15]). In two dimensional case the class BV of functions of bounded variation was introduced by Hardy [8].

Let  $f$  be a real function of two variable of period  $2\pi$  with respect to each variable. Given intervals  $I = (a, b)$ ,  $J = (c, d)$  and points  $x, y$  from  $T := [0, 2\pi]$  we denote

$$f(I, y) := f(b, y) - f(a, y), \quad f(x, J) = f(x, d) - f(x, c)$$

and

$$f(I, J) := f(a, c) - f(a, d) - f(b, c) + f(b, d).$$

Let  $E = \{I_i\}$  be a collection of nonoverlapping intervals from  $T$  ordered in arbitrary way and let  $\Omega$  be the set of all such collections  $E$ .

The Hardy class BV consists of functions  $f$  satisfying the condition

$$\sup_{E \in \Omega} \sum_i |f(I_i, 0)| + \sup_x \sup_{F \in \Omega} \sum_j |f(0, J_j)| + \sup_{F, E \in \Omega} \sum_i \sum_j |f(I_i, J_j)| < \infty,$$

where  $E = \{I_i\}$  and  $F = \{J_j\}$ .

In [6] U. Goginava introduced the class  $PBV$  of functions of bounded partial bounded variation, i.e. functions  $f$  having uniformly bounded variation

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with respect to each variable:

$$\sup_y \sup_{E \in \Omega} \sum_i |f(I_i, y)| + \sup_x \sup_{F \in \Omega} \sum_j |f(x, J_j)| < \infty.$$

For the sequence of positive numbers  $\Lambda = \{\lambda_n\}_{n=1}^\infty$  we denote

$$\Lambda V_1(f) = \sup_y \sup_{E \in \Omega} \sum_i \frac{|f(I_i, y)|}{\lambda_i} \quad (E = \{I_i\}),$$

$$\Lambda V_2(f) = \sup_x \sup_{F \in \Omega} \sum_j \frac{|f(x, J_j)|}{\lambda_j} \quad (F = \{J_j\}),$$

$$\Lambda V_{1,2}(f) = \sup_{F, E \in \Omega} \sum_i \sum_j \frac{|f(I_i, J_j)|}{\lambda_i \lambda_j}.$$

**Definition 1.** We say that the function  $f$  has Bounded  $\Lambda$ -variation on  $T^2 = [0, 2\pi]^2$  and write  $f \in \Lambda BV$ , if

$$\Lambda V(f) := \Lambda V_1(f) + \Lambda V_2(f) + \Lambda V_{1,2}(f) < \infty.$$

We say that the function  $f$  has Bounded Partial  $\Lambda$ -variation and write  $f \in P\Lambda BV$  if

$$P\Lambda V(f) := \Lambda V_1(f) + \Lambda V_2(f) < \infty.$$

If  $\lambda_n \equiv 1$  (or if  $0 < c < \lambda_n < C < \infty$ ,  $n = 1, 2, \dots$ ) the classes  $\Lambda BV$  and  $P\Lambda BV$  coincide with the Hardy class  $BV$  and  $PBV$  respectively. Hence it is reasonable to assume that  $\lambda_n \rightarrow \infty$  and since the intervals in  $E = \{I_i\}$  are ordered arbitrarily, we will suppose, without loss of generality, that the sequence  $\{\lambda_n\}$  is increasing. Thus,

$$(1) \quad 1 < \lambda_1 \leq \lambda_2 \leq \dots, \quad \lim_{n \rightarrow \infty} \lambda_n = \infty.$$

In the case when  $\lambda_n = n$ ,  $n = 1, 2, \dots$  we say *Harmonic Variation* instead of  $\Lambda$ -variation and write  $H$  instead of  $\Lambda$  ( $HBV$ ,  $PHBV$ ,  $HV(f)$ , etc).

The notion of  $\Lambda$ -variation was introduced by D. Waterman [13] in one dimensional case and A. Sahakian [12] in two dimensional case.

**Definition 2** (Waterman [14]). Let  $\Lambda = \{\lambda_n\}_{n=1}^\infty$  and  $\Lambda_k = \{\lambda_n\}_{n=k}^\infty$ ,  $k = 1, 2, \dots$ . We say that the function  $f$  is continuous in  $\Lambda$ -variation and write  $f \in C\Lambda BV$ , if

$$\lim_{k \rightarrow \infty} \Lambda_k V(f) = 0.$$

## 2. $(C; \alpha, \beta)$ $(-1 < \alpha, \beta < 0)$ SUMMABILITY

Let  $f \in L^1(T^2)$ . The Fourier series of  $f$  with respect to the trigonometric system is the series

$$S[f, (x, y)] := \sum_{m, n=-\infty}^{+\infty} \widehat{f}(m, n) e^{imx} e^{iny},$$

where

$$\widehat{f}(m, n) = \frac{1}{4\pi^2} \int_0^{2\pi} \int_0^{2\pi} f(x, y) e^{-imx} e^{-iny} dx dy$$

are the Fourier coefficients of the function  $f$ . The rectangular partial sums are defined as follows:

$$S_{M,N}[f, (x, y)] := \sum_{m=-M}^M \sum_{n=-N}^N \widehat{f}(m, n) e^{imx} e^{iny},$$

The Cesàro  $(C; \alpha, \beta)$ ,  $\alpha, \beta > -1$ , means of two-dimensional Fourier series are defined by

$$\sigma_{n,m}^{\alpha,\beta} f(x, y) := \frac{1}{A_n^\alpha} \frac{1}{A_m^\beta} \sum_{i=0}^n \sum_{j=0}^m A_{n-i}^{\alpha-1} A_{m-j}^{\beta-1} S_{i,j}[f, (x, y)]$$

where

$$A_0^\alpha = 1, \quad A_k^\alpha = \frac{(\alpha+1) \cdots (\alpha+k)}{k!}, \quad k = 1, 2, \dots$$

We say that the double trigonometric Fourier series of the function  $f$  is  $(C; -\alpha, -\beta)$  summable to  $f$ , if

$$\lim_{n, m \rightarrow \infty} \sigma_{n,m}^{\alpha,\beta} f(x, y) = f(x, y).$$

It is well-known that (see [17], p. 157 )

$$\sigma_{mn}^{(\alpha,\beta)} f(x, y) = \frac{1}{\pi^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} f(x+t, y+s) K_m^\alpha(s) K_n^\beta(t) ds dt,$$

where the kernel  $K_n^\alpha$ ,  $-1 < \alpha < 0$  satisfies the following conditions:

$$(2) \quad |K_n^{-\alpha}(u)| \leq 2n, \quad u \in T,$$

$$(3) \quad K_n^\alpha(u) = \varphi_n^\alpha(u) + O(1/nt^2), \quad 0 \leq |u| \leq \pi,$$

where

$$(4) \quad \varphi_n^\alpha(u) = \frac{\sin[(n+1/2+\alpha/2)u - \alpha\pi/2]}{A_n^\alpha [2 \sin u/2]^{1+\alpha}},$$

The coefficients  $A_n^\alpha$  have following bounds:

$$(5) \quad c_1(\alpha)n^\alpha \leq A_n^\alpha \leq c_2(\alpha)n^\alpha.$$

Denote

$$\begin{aligned} {}_1\Delta_i^m f(x, y) &:= f\left(x + \frac{2i\pi}{m}, y\right) - f\left(x + \frac{(2i+1)\pi}{m}, y\right), \\ {}_2\Delta_j^n f(x, y) &:= f\left(x, y + \frac{2j\pi}{n}\right) - f\left(x, y + \frac{(2j+1)\pi}{n}\right), \\ \Delta_{ij}^{mn} f(x, y) &= f\left(x + \frac{2i\pi}{m}, y + \frac{2j\pi}{n}\right) - f\left(x + \frac{(2i+1)\pi}{m}, y + \frac{2j\pi}{n}\right) \end{aligned}$$

$$-f\left(x + \frac{2i\pi}{m}, y + \frac{(2j+1)\pi}{n}\right) + f\left(x + \frac{(2i+1)\pi}{m}, y + \frac{(2j+1)\pi}{n}\right).$$

### 3. FORMULATION OF PROBLEMS

Let  $C(T^2)$  be the space of  $2\pi$ -periodic with respect to each variable continuous functions with the norm

$$\|f\|_C := \sup_{x,y \in T^2} |f(x,y)|.$$

For the function  $f(x,y)$  we denote by  $f(x \pm 0, y \pm 0)$  the open coordinate quadrant limits (if exist) at the point  $(x,y)$  and set

$$(6) \quad \sum f(x \pm 0, y \pm 0) \\ = \{f(x+0, y+0) + f(x+0, y-0) + f(x-0, y+0) + f(x-0, y-0)\}.$$

The well known Dirichlet-Jordan theorem (see [17]) states that the Fourier series of a function  $f(x)$ ,  $x \in T$  of bounded variation converges at every point  $x$  to the value  $[f(x+0) + f(x-0)]/2$ . If  $f$  is in addition continuous on  $T$  the Fourier series converges uniformly on  $T$ . This result was generalized by Waterman [13].

**Theorem W1** (Waterman [13]). *If  $f \in HBV$ , then  $S[f]$  converges at every point  $x$  to the value  $[f(x+0) + f(x-0)]/2$ . If  $f$  is in addition continuous on  $T$ , then  $S[f]$  converges uniformly on  $T$ .*

Hardy [8] generalized the Dirichlet-Jordan theorem to the double Fourier series. He proved that if function  $f(x,y)$  has bounded variation in the sense of Hardy ( $f \in BV$ ), then  $S[f]$  converges at any point  $(x,y)$  to the value  $\frac{1}{4} \sum f(x \pm 0, y \pm 0)$ . If  $f$  is in addition continuous on  $T^2$  then  $S[f]$  converges uniformly on  $T^2$ .

**Theorem S** (Sahakian [12]). *The Fourier series of a function  $f(x,y) \in HBV$  converges to  $\frac{1}{4} \sum f(x \pm 0, y \pm 0)$  at any point  $(x,y)$ , where the quadrant limits (6) exist. The convergence is uniformly on any compact  $K$ , where the function  $f$  is continuous.*

Analog of Theorem S for higher dimensions can be found in [11] and [1]. Convergence of spherical and other partial sums of double Fourier series of functions of bounded  $\Lambda$ -variation was investigated in details by Dyachenko (see [3], [4], [5] and references therein).

The first author [6] has proved that in Hardy's theorem there is no need to require the boundedness of mixed variation. In particular, the following is true

**Theorem G1** (Goginava [6]). *Let  $f \in C(T^2) \cap PBV$ . Then  $S[f]$  converges uniformly on  $T^2$ .*

For one-dimensional Fourier series Waterman [14] proved the following

**Theorem W2** (Waterman [14]). *Let  $0 < \alpha < 1$  and  $f \in C\{n^{1-\alpha}\}BV$ . Then  $S[f]$  is everywhere  $(C, -\alpha)$  summable to the value  $[f(x+0) + f(x-0)]/2$  and the summability is uniform on each closed interval of continuity.*

Later Sablin proved in [11], that for  $0 < \alpha < 1$  the classes  $\{n^{1-\alpha}\}BV$  and  $C\{n^{1-\alpha}\}BV$  coincide.

Zhizhiashvili [16] has investigated the convergence of Cesàro means of double trigonometric Fourier series. In particular, the following theorem was proved.

**Theorem Zh** (Zhizhiashvili [16]). *Let  $\alpha, \beta > 0$  and  $\alpha + \beta < 1$ . If  $f \in BV$ , then the double Fourier series of  $f$  is  $(C; -\alpha, -\beta)$  summable to  $\frac{1}{4} \sum f(x \pm 0, y \pm 0)$  in any point  $(x, y)$ . The convergence is uniformly on any compact  $K$ , where the function  $f$  is continuous.*

For functions of partial bounded variation the problem was solved by the first author in [7].

**Theorem G2** (Goginava [7]). *Let  $f \in C(T^2) \cap PBV$  and  $\alpha + \beta < 1$ ,  $\alpha, \beta > 0$ . Then the double trigonometric Fourier series of the function  $f$  is uniformly  $(C; -\alpha, -\beta)$  summable to  $f$ .*

**Theorem G3** (Goginava [7]). *Let  $\alpha + \beta \geq 1$ ,  $\alpha, \beta > 0$ . Then there exists a continuous function  $f_0 \in PBV$  such that the Cesàro  $(C; -\alpha, -\beta)$  means  $\sigma_{n,m}^{-\alpha,-\beta}(f_0; 0, 0)$  of the double trigonometric Fourier series of  $f_0$  diverge over cubes.*

In this paper we consider the following problem. Let  $\alpha, \beta \in (0, 1)$ ,  $\alpha + \beta < 1$ .  
1. Under what conditions on the sequence  $\Lambda = \{\lambda_n\}$  the double Fourier series of the function  $f \in PABV$  is  $(C; -\alpha, -\beta)$  summable.

The solution is given in Theorems 1 and 2 bellow.

#### 4. MAIN RESULTS

**Theorem 1.** *Let  $\alpha, \beta \in (0, 1)$ ,  $\alpha + \beta < 1$  and the sequence  $\Lambda = \{\lambda_k\}$  satisfies the conditions:*

$$\frac{\lambda_k}{k^{1-(\alpha+\beta)}} \downarrow 0, \quad \sum_{k=1}^{\infty} \frac{\lambda_k}{k^{2-(\alpha+\beta)}} < \infty.$$

*Then the double Fourier series of the function  $f \in PABV$  is  $(C; -\alpha, -\beta)$  summable to  $\frac{1}{4} \sum f(x \pm 0, y \pm 0)$  at any point  $(x, y)$ , where the quadrant limits (6) exist. The convergence is uniform on any compact  $K$ , where the function  $f$  is continuous.*

**Theorem 2.** *Let  $\alpha, \beta \in (0, 1)$ ,  $\alpha + \beta < 1$  and the sequence  $\Lambda = \{\lambda_k\}$  satisfies the conditions:*

$$\frac{\lambda_k}{k^{1-(\alpha+\beta)}} \downarrow 0, \quad \sum_{k=1}^{\infty} \frac{\lambda_k}{k^{2-(\alpha+\beta)}} = \infty.$$

Then there exists a continuous function  $f \in P\Lambda BV$  for which  $(C; -\alpha, -\beta)$  means of two-dimensional Fourier series diverges over cubes at  $(0, 0)$ .

**Corollary 1.** Let  $\alpha, \beta \in (0, 1)$ ,  $\alpha + \beta < 1$ .

a) If  $f \in P\left\{\frac{n^{1-(\alpha+\beta)}}{\log^{1+\varepsilon}(n+1)}\right\}BV$  for some  $\varepsilon > 0$ , then the double Fourier series of the function  $f$  is  $(C; -\alpha, -\beta)$  summable to  $\frac{1}{4} \sum f(x \pm 0, y \pm 0)$  in any point  $(x, y)$ , where the quadrant limits (6) exist. The convergence is uniform on any compact  $K$ , where the function  $f$  is continuous.

b) There exists a continuous function  $f \in P\left\{\frac{n^{1-(\alpha+\beta)}}{\log(n+1)}\right\}BV$  such that  $(C; -\alpha, -\beta)$  means of two-dimensional Fourier series of  $f$  diverges over cubes at  $(0, 0)$ .

**Corollary 2.** Let  $\alpha, \beta \in (0, 1)$ ,  $\alpha + \beta < 1$  and  $f \in PBV$ . Then the double Fourier series of the function  $f$  is  $(C; -\alpha, -\beta)$  summable to  $\frac{1}{4} \sum f(x \pm 0, y \pm 0)$  in any point  $(x, y)$ , where the quadratic limits (6) exist. The convergence is uniform on any compact  $K$ , where the function  $f$  is continuous.

## 5. PROOFS

*Proof of Theorem 1.* It is easy to show that

$$\begin{aligned} & \sigma_{mn}^{(-\alpha, -\beta)} f(x, y) - \frac{1}{4} \sum f(x \pm 0, y \pm 0) \\ &= \frac{1}{\pi^2} \sum_{i=1}^4 \int_0^\pi \int_0^\pi \varphi_i(x, y, s, t) K_m^{-\alpha}(s) K_n^{-\beta}(t) ds dt \\ &=: \sum_{i=1}^4 I_{mn}^{(k)}(x, y). \end{aligned}$$

where

$$\begin{aligned} \varphi_1(x, y, s, t) &:= f(x+s, y+t) - f(x+0, y+0), \\ \varphi_2(x, y, s, t) &:= f(x-s, y+t) - f(x-0, y+0), \\ \varphi_3(x, y, s, t) &:= f(x+s, y-t) - f(x+0, y-0), \\ \varphi_4(x, y, s, t) &:= f(x-s, y-t) - f(x-0, y-0). \end{aligned}$$

For  $I_{mn}^{(1)}(x, y)$  we can write

$$\begin{aligned} (7) \quad & \pi^2 I_{mn}^{(1)}(x, y) \\ &= \left( \int_0^{\pi/m} \int_0^{\pi/n} + \int_0^{\pi/m} \int_{\pi/n}^\pi + \int_{\pi/m}^\pi \int_0^{\pi/n} + \int_{\pi/m}^\pi \int_{\pi/n}^\pi \right) \left( \varphi_1(x, y, s, t) K_m^{-\alpha}(s) K_n^{-\beta}(t) ds dt \right) \\ &=: \sum_{k=1}^4 I_{mn}^{(1k)}(x, y). \end{aligned}$$

From (2) we have

$$(8) \quad \begin{aligned} \left| I_{mn}^{(11)}(x, y) \right| &\leq c(\alpha, \beta) mn \int_0^{\pi/m} \int_0^{\pi/n} |\varphi_1(x, y, s, t)| ds dt \\ &\leq c(\alpha, \beta) \sup_{0 < s < \pi/m, 0 < t < \pi/n} |\varphi_1(x, y, s, t)| = o(1) \quad \text{as } m, n \rightarrow \infty. \end{aligned}$$

Using (3), we obtain

$$(9) \quad \begin{aligned} I_{mn}^{(12)}(x, y) &= \int_0^{\pi/m} \int_{\pi/n}^{\pi} \varphi_1(x, y, s, t) K_m^{-\alpha}(s) \varphi_n^{-\beta}(t) ds dt \\ &\quad + \int_0^{\pi/m} \int_{\pi/n}^{\pi} \varphi_1(x, y, s, t) K_m^{-\alpha}(s) O\left(\frac{1}{nt^2}\right) ds dt \\ &=: I_{mn}^{(121)}(x, y) + I_{mn}^{(122)}(x, y). \end{aligned}$$

We can write

$$(10) \quad \begin{aligned} \left| I_{mn}^{(122)}(x, y) \right| &\leq \int_0^{\pi/m} \int_{\pi/\sqrt{n}}^{\pi/\sqrt{n}} |\varphi_1(x, y, s, t)| |K_m^{-\alpha}(s)| O\left(\frac{1}{nt^2}\right) ds dt \\ &\quad + \int_0^{\pi/m} \int_{\pi/\sqrt{n}}^{\pi} |\varphi_1(x, y, s, t)| |K_m^{-\alpha}(s)| O\left(\frac{1}{nt^2}\right) ds dt \\ &\leq c(\alpha, \beta, f) \left\{ \sup_{0 < s < \pi/m, 0 < t < \pi/\sqrt{n}} |\varphi_1(x, y, s, t)| + \int_{\pi/\sqrt{n}}^{\pi} \frac{dt}{nt^2} \right\} \\ &= o(1) \quad \text{as } n, m \rightarrow \infty. \end{aligned}$$

In order to estimate  $I_{mn}^{(121)}(x, y)$  it is enough to estimate the following expression

$$J_{mn}(x, y) := n^\beta \int_0^{\pi/m} \int_{\pi/n}^{\pi} \varphi_1(x, y, s, t) K_m^{-\alpha}(s) w_\beta(t) \sin ntds dt,$$

where

$$w_\beta(t) = \frac{\cos \frac{1-\beta}{2}t}{(\sin t/2)^{1-\beta}}.$$

We have

$$\begin{aligned}
J_{mn}(x, y) &= n^\beta \sum_{i=1}^{n-1} \int_0^{\pi/m} K_m^{-\alpha}(s) \left( \int_{i\pi/n}^{(i+1)\pi/n} \varphi_1(x, y, s, t) w_\beta(t) \sin ntdt \right) ds \\
&= n^\beta \sum_{i=1}^{(n-1)/2} \int_0^{\pi/m} K_m^{-\alpha}(s) \left( \int_0^{\pi/n} \left[ \varphi_1\left(x, y, s, t + \frac{2i\pi}{n}\right) - \varphi_1\left(x, y, s, t + \frac{(2i+1)\pi}{n}\right) \right] \right. \\
&\quad \left. w_\beta\left(t + \frac{2i\pi}{n}\right) \sin ntdt \right) ds \\
&\quad + n^\beta \sum_{i=1}^{(n-1)/2} \int_0^{\pi/m} K_m^{-\alpha}(s) \left( \int_0^{\pi/n} \varphi_1\left(x, y, s, t + \frac{(2i+1)\pi}{n}\right) \right. \\
&\quad \left. \left[ w_\beta\left(t + \frac{2i\pi}{n}\right) - w_\beta\left(t + \frac{(2i+1)\pi}{n}\right) \right] \sin ntdt \right) ds \\
&=: J_{mn}^{(1)}(x, y) + J_{mn}^{(2)}(x, y).
\end{aligned}$$

Using the following inequality:

$$(11) \quad \left| w_\beta\left(t + \frac{2i\pi}{n}\right) - w_\beta\left(t + \frac{(2i+1)\pi}{n}\right) \right| \leq \frac{c(\beta) n^{1-\beta}}{i^{2-\beta}},$$

for  $J_{mn}^{(2)}(x, y)$  we can write

$$\begin{aligned}
(12) \quad & \left| J_{mn}^{(2)}(x, y) \right| \\
& \leq c(\beta) mn \sum_{i=1}^{(n-1)/2} \frac{1}{i^{2-\beta}} \int_0^{\pi/m} \left( \int_0^{\pi/n} \left| \varphi_1\left(x, y, s, t + \frac{(2i+1)\pi}{n}\right) \right| dt \right) ds \\
& \leq c(\beta) nm \sum_{i \leq \sqrt{n}} \frac{1}{i^{2-\beta}} \int_0^{\pi/m} \left( \int_0^{\pi/n} \left| \varphi_1\left(x, y, s, t + \frac{(2i+1)\pi}{n}\right) \right| dt \right) ds \\
& + c(\beta) nm \sum_{\sqrt{n} < i \leq (n-1)/2} \frac{1}{i^{2-\beta}} \int_0^{\pi/m} \left( \int_0^{\pi/n} \left| \varphi_1\left(x, y, s, t + \frac{(2i+1)\pi}{n}\right) \right| dt \right) ds \\
& \leq c(\beta) \sup_{0 < s < \pi/n, 0 < s < 4\pi/\sqrt{n}} |\varphi_1(x, y, s, t)| + c(\beta, f) \left( \frac{1}{\sqrt{n}} \right)^{1-\beta} = o(1),
\end{aligned}$$

as  $n, m \rightarrow \infty$ .

To estimate  $J_{mn}^{(1)}(x, y)$ , we denote

$$(13) \quad \mu(n, m) := \left[ \min \left\{ \frac{1}{2} \ln n - 1, (s(n, m))^{-1} \right\} \right],$$



where  $[a]$  is the integer part of  $a$  and

$$(14) \quad s(n, m) := \sup_{0 < s < \pi/m, 0 < t < \pi \ln n/n} |\varphi_1(x, y, s, t)|.$$

Then we have

$$\begin{aligned}
 (15) \quad & \left| J_{mn}^{(1)}(x, y) \right| \leq c(\beta) nm \int_0^{\pi/m} \left( \int_0^{\pi/n} \sum_{i=1}^{\mu(n, m)} \frac{1}{i^{1-\beta}} \left| \varphi_1 \left( x, y, s, t + \frac{2i\pi}{n} \right) \right. \right. \\
 & \quad \left. \left. - \varphi_1 \left( x, y, s, t + \frac{(2i+1)\pi}{n} \right) \right| dt ds \right) \\
 & + c(\beta) nm \int_0^{\pi/m} \left( \int_0^{\pi/n} \sum_{i=\mu(n, m)}^{(n-1)/2} \frac{1}{i^{1-\beta}} \left| \varphi_1 \left( x, y, s, t + \frac{2i\pi}{n} \right) \right. \right. \\
 & \quad \left. \left. - \varphi_1 \left( x, y, s, t + \frac{(2i+1)\pi}{n} \right) \right| dt \right) ds \\
 & \leq c(\beta) \sup_{0 < s < \pi/m, 0 < t < (2\mu(n, m)+1)\pi/n} |\varphi_1(x, y, s, t)| (\mu(n, m))^\beta \\
 & + c(\beta) nm \int_0^{\pi/m} \left( \int_0^{\pi/n} \sum_{i=\mu(n, m)}^{(n-1)/2} \frac{1}{\lambda_i} \left| \varphi_1 \left( x, y, s, t + \frac{2i\pi}{n} \right) \right. \right. \\
 & \quad \left. \left. - \varphi_1 \left( x, y, s, t + \frac{(2i+1)\pi}{n} \right) \right| \frac{\lambda_i}{i^{1-\beta}} dt \right) ds \\
 & \leq c(\beta) \sup_{0 < s < \pi/m, 0 < t < \pi \ln n/n} |\varphi_1(x, y, s, t)| (\mu(n, m))^\beta \\
 & + c(\beta) nm \frac{\lambda_{\mu(n, m)}}{(\mu(n, m))^{1-\beta}} \int_0^{\pi/m} \left( \int_0^{\pi/n} \sum_{i=\mu(n, m)}^{(n-1)/2} \frac{1}{\lambda_i} \left| \varphi_1 \left( x, y, s, t + \frac{2i\pi}{n} \right) \right. \right. \\
 & \quad \left. \left. - \varphi_1 \left( x, y, s, t + \frac{(2i+1)\pi}{n} \right) \right| dt \right) ds \\
 & \leq c(\beta) s(n, m) (\mu(n, m))^\beta + c(\beta) \frac{\lambda_{\mu(n, m)}}{(\mu(n, m))^{1-\beta}} V_2 \Lambda(f) = o(1) \quad \text{as } n, m \rightarrow \infty.
 \end{aligned}$$

Combining (9), (10), (12) and (15) we conclude that

$$(16) \quad I_{mn}^{(12)}(x, y) \rightarrow 0 \quad \text{as } m, n \rightarrow \infty.$$

Analogously, we can prove that

$$(17) \quad I_{mn}^{(13)}(x, y) \rightarrow 0 \quad \text{as } m, n \rightarrow \infty.$$

In order to estimate  $I_{mn}^{(14)}(x, y)$  it is enough to estimate the following expression

$$L_{mn}(x, y) := m^\alpha n^\beta \int_{\pi/m}^{\pi} \int_{\pi/n}^{\pi} \varphi_1(x, y, s, t) w_\alpha(s) w_\beta(t) \sin ms \sin ntdtds.$$

We have

$$\begin{aligned}
(18) \quad L_{mn}(x, y) &= m^\alpha n^\beta \sum_{i=1}^{(m-1)/2} \sum_{j=1}^{(n-1)/2} \int_0^{\pi/m} \int_0^{\pi/n} \varphi_1\left(x, y, s + \frac{2i\pi}{m}, t + \frac{2j\pi}{n}\right) \\
&\quad \times w_\alpha\left(s + \frac{2i\pi}{m}\right) w_\beta\left(t + \frac{2j\pi}{n}\right) \sin ms \sin ntdtds \\
&\quad - m^\alpha n^\beta \sum_{i=1}^{(m-1)/2} \sum_{j=1}^{(n-1)/2} \int_0^{\pi/m} \int_0^{\pi/n} \varphi_1\left(x, y, s + \frac{(2i+1)\pi}{m}, t + \frac{2j\pi}{n}\right) \\
&\quad \times w_\alpha\left(s + \frac{(2i+1)\pi}{m}\right) w_\beta\left(t + \frac{2j\pi}{n}\right) \sin ms \sin ntdtds \\
&\quad - m^\alpha n^\beta \sum_{i=1}^{(m-1)/2} \sum_{j=1}^{(n-1)/2} \int_0^{\pi/m} \int_0^{\pi/n} \varphi_1\left(x, y, s + \frac{2i\pi}{m}, t + \frac{(2j+1)\pi}{n}\right) \\
&\quad \times w_\alpha\left(s + \frac{2i\pi}{m}\right) w_\beta\left(t + \frac{(2j+1)\pi}{n}\right) \sin ms \sin ntdtds \\
&\quad + m^\alpha n^\beta \sum_{i=1}^{(m-1)/2} \sum_{j=1}^{(n-1)/2} \int_0^{\pi/m} \int_0^{\pi/n} \varphi_1\left(x, y, s + \frac{(2i+1)\pi}{m}, t + \frac{(2j+1)\pi}{n}\right) \\
&\quad \times w_\alpha\left(s + \frac{(2i+1)\pi}{m}\right) w_\beta\left(t + \frac{(2j+1)\pi}{n}\right) \sin ms \sin ntdtds \\
&= m^\alpha n^\beta \sum_{i=1}^{(m-1)/2} \sum_{j=1}^{(n-1)/2} \int_0^{\pi/m} \int_0^{\pi/n} \left[ \varphi_1\left(x, y, s + \frac{2i\pi}{m}, t + \frac{2j\pi}{n}\right) \right. \\
&\quad \left. - \varphi_1\left(x, y, s + \frac{(2i+1)\pi}{m}, t + \frac{2j\pi}{n}\right) - \varphi_1\left(x, y, s + \frac{2i\pi}{m}, t + \frac{(2j+1)\pi}{n}\right) \right. \\
&\quad \left. + \varphi_1\left(x, y, s + \frac{(2i+1)\pi}{m}, t + \frac{(2j+1)\pi}{n}\right) \right] \\
&\quad \times w_\alpha\left(s + \frac{2i\pi}{m}\right) w_\beta\left(t + \frac{2j\pi}{n}\right) \sin ms \sin ntdtds \\
&\quad + m^\alpha n^\beta \sum_{i=1}^{(m-1)/2} \sum_{j=1}^{(n-1)/2} \int_0^{\pi/m} \int_0^{\pi/n} \left[ \varphi_1\left(x, y, s + \frac{(2i+1)\pi}{m}, t + \frac{2j\pi}{n}\right) \right.
\end{aligned}$$

$$\begin{aligned}
& -\varphi_1 \left( x, y, s + \frac{(2i+1)\pi}{m}, t + \frac{(2j+1)\pi}{n} \right) \Big] \\
& \times \left[ w_\alpha \left( s + \frac{2i\pi}{m} \right) - w_\alpha \left( s + \frac{(2i+1)\pi}{m} \right) \right] w_\beta \left( t + \frac{2j\pi}{n} \right) \sin ms \sin ntdtds \\
& + m^\alpha n^\beta \sum_{i=1}^{(m-1)/2} \sum_{j=1}^{(n-1)/2} \int_0^{\pi/m} \int_0^{\pi/n} \left[ \varphi_1 \left( x, y, s + \frac{2i\pi}{m}, t + \frac{(2j+1)\pi}{n} \right) \right. \\
& \quad \left. - \varphi_1 \left( x, y, s + \frac{(2i+1)\pi}{m}, t + \frac{(2j+1)\pi}{n} \right) \right] \\
& \times \left[ w_\beta \left( t + \frac{2j\pi}{n} \right) - w_\beta \left( t + \frac{(2j+1)\pi}{n} \right) \right] w_\alpha \left( s + \frac{2i\pi}{m} \right) \sin ms \sin ntdtds \\
& + m^\alpha n^\beta \sum_{i=1}^{(m-1)/2} \sum_{j=1}^{(n-1)/2} \int_0^{\pi/m} \int_0^{\pi/n} \varphi_1 \left( x, y, s + \frac{(2i+1)\pi}{m}, t + \frac{(2j+1)\pi}{n} \right) \\
& \quad \times \left[ w_\beta \left( t + \frac{2j\pi}{n} \right) - w_\beta \left( t + \frac{(2j+1)\pi}{n} \right) \right] \\
& \quad \left[ w_\alpha \left( s + \frac{2i\pi}{m} \right) - w_\alpha \left( s + \frac{(2i+1)\pi}{m} \right) \right] \sin ms \sin ntdtds \\
& =: \sum_{k=1}^4 L_{mn}^{(k)}(x, y).
\end{aligned}$$

By (11) we obtain

$$\begin{aligned}
(19) \quad & \left| L_{mn}^{(4)}(x, y) \right| \leq c(\alpha, \beta) mn \sum_{i=1}^{\lfloor \sqrt{m} \rfloor} \frac{1}{i^{2-\alpha}} \sum_{j=1}^{\lfloor \sqrt{n} \rfloor} \frac{1}{j^{2-\beta}} \\
& \int_0^{\pi/m} \int_0^{\pi/n} \left| \varphi_1 \left( x, y, s + \frac{(2i+1)\pi}{m}, t + \frac{(2j+1)\pi}{n} \right) \right| \\
& + c(\alpha, \beta, f) \sum_{i=1}^{\infty} \frac{1}{i^{2-\alpha}} \sum_{j=\lfloor \sqrt{n} \rfloor}^{\infty} \frac{1}{j^{2-\beta}} \\
& + c(\alpha, \beta, f) mn \sum_{i=\lfloor \sqrt{m} \rfloor}^{\infty} \frac{1}{i^{2-\alpha}} \sum_{j=1}^{\infty} \frac{1}{j^{2-\beta}} \\
& \leq c(\alpha, \beta) \sup_{0 < s < 4\pi/\sqrt{m}, 0 < t < 4\pi/\sqrt{n}} |\varphi_1(x, y, s, t)| + o(1) \\
& = o(1) \quad \text{as } n, m \rightarrow \infty.
\end{aligned}$$

Let

$$(20) \quad \tau(n, m) := \left[ \min \left\{ \frac{1}{2} \ln n - 1, \frac{1}{2} \ln m - 1, (l(n, m))^{-1} \right\} \right],$$

where

$$l(n, m) := \sup_{0 < s < \pi \ln m/m, 0 < t < \pi \ln n/n} |\varphi_1(x, y, s, t)|$$

Then we can write

$$\begin{aligned}
(21) \quad & \left| L_{mn}^{(3)}(x, y) \right| \\
& \leq c(\alpha, \beta) mn \sum_{i=1}^{\tau(n, m)} \frac{1}{i^{1-\alpha}} \sum_{j=1}^{\tau(n, m)} \frac{1}{j^{2-\beta}} \int_0^{\pi/m} \int_0^{\pi/n} \left| \varphi_1 \left( x, y, s + \frac{2i\pi}{m}, t + \frac{(2j+1)\pi}{n} \right) \right. \\
& \quad \left. - \varphi_1 \left( x, y, s + \frac{(2i+1)\pi}{m}, t + \frac{(2j+1)\pi}{n} \right) \right| dt ds \\
& + c(\alpha, \beta) mn \sum_{i=\tau(n, m)}^{(m-1)/2} \frac{1}{\lambda_i} \frac{\lambda_i}{i^{1-\alpha}} \sum_{j=1}^{\tau(n, m)} \frac{1}{j^{2-\beta}} \int_0^{\pi/m} \int_0^{\pi/n} \left| \varphi_1 \left( x, y, s + \frac{2i\pi}{m}, t + \frac{(2j+1)\pi}{n} \right) \right. \\
& \quad \left. - \varphi_1 \left( x, y, s + \frac{(2i+1)\pi}{m}, t + \frac{(2j+1)\pi}{n} \right) \right| dt ds \\
& + c(\alpha, \beta) mn \sum_{i=1}^{(m-1)/2} \frac{1}{\lambda_i} \frac{\lambda_i}{i^{1-\alpha}} \sum_{j=\tau(n, m)}^{(n-1)/2} \frac{1}{j^{2-\beta}} \int_0^{\pi/m} \int_0^{\pi/n} \left| \varphi_1 \left( x, y, s + \frac{2i\pi}{m}, t + \frac{(2j+1)\pi}{n} \right) \right. \\
& \quad \left. - \varphi_1 \left( x, y, s + \frac{(2i+1)\pi}{m}, t + \frac{(2j+1)\pi}{n} \right) \right| dt ds \\
& \leq c(\alpha, \beta) l(n, m) (\tau(n, m))^{\alpha+\beta} \\
& + c(\alpha, \beta) \frac{\lambda_{\tau(n, m)}}{(\tau(n, m))^{1-\alpha}} V_1 \Lambda(f) + c(\alpha, \beta) \frac{1}{(\tau(n, m))^{1-\beta}} V_1 \Lambda(f) \\
& = o(1) \text{ as } n, m \rightarrow \infty.
\end{aligned}$$

Analogously, we can prove that

$$(22) \quad \left| L_{mn}^{(2)}(x, y) \right| = o(1) \text{ as } n, m \rightarrow \infty.$$

For  $L_{mn}^{(1)}(x, y)$  we can write

$$\begin{aligned}
(23) \quad & \left| L_{mn}^{(1)}(x, y) \right| \\
& \leq c(\alpha, \beta) mn \left\{ \sum_{i=1}^{\tau(n, m)} \frac{1}{i^{1-\alpha}} \sum_{j=1}^{\tau(n, m)} \frac{1}{j^{1-\beta}} + \sum_{i=\tau(n, m)}^{(m-1)/2} \frac{1}{i^{1-\alpha}} \sum_{j=1}^{\tau(n, m)} \frac{1}{j^{1-\beta}} \right. \\
& \quad \left. + \sum_{i=1}^{\tau(n, m)} \frac{1}{i^{1-\alpha}} \sum_{j=\tau(n, m)}^{(n-1)/2} \frac{1}{j^{1-\beta}} + \sum_{i=\tau(n, m)}^{(m-1)/2} \frac{1}{i^{1-\alpha}} \sum_{j=\tau(n, m)}^{(n-1)/2} \frac{1}{j^{1-\beta}} \right\} \\
& \quad \left( \int_0^{\pi/m} \int_0^{\pi/n} |\Delta_{ij}^{mn} f(x + s, y + t)| ds dt \right) =: \sum_{k=1}^4 L_{mn}^{(1k)}(x, y).
\end{aligned}$$

From (20) we obtain that

$$(24) \quad \left| L_{mn}^{(11)}(x, y) \right| \leq c(\alpha, \beta) \left( l(n, m)^{1-\alpha-\beta} \right) \rightarrow 0 \text{ as } m, n \rightarrow \infty.$$

Next, we have

$$(25) \quad \left| L_{mn}^{(13)}(x, y) \right| \leq c(\alpha, \beta) mn \left\{ \sum_{i=1}^{\tau(n, m)} \frac{1}{i^{1-\alpha}} \sum_{j=\tau(n, m)}^{(n-1)/2} \frac{1}{j^{1-\beta}} \right. \\ \left. \left( \int_0^{\pi/m} \int_0^{\pi/n} |\Delta_{ij}^{mn} f(x+s, y+t)| ds dt \right) \right\} \\ \leq c(\alpha, \beta) n \left\{ \sum_{i=1}^{\tau(n, m)} \frac{1}{i^{1-\alpha}} \left( \int_0^{\pi/n} \sup_x \sum_{j=\tau(n, m)}^{(n-1)/2} \frac{\lambda_j}{j^{1-\beta}} \frac{1}{\lambda_j} |{}_2\Delta_j^n f(x, y+t)| dt \right) \right\} \\ \leq c(\alpha, \beta) \frac{\tau(n, m)}{(\tau(n, m))^{1-\beta-\alpha}} V_2 \Lambda(f) \rightarrow 0, \text{ as } n, m \rightarrow \infty.$$

Analogously, we can prove that

$$(26) \quad \left| L_{mn}^{(12)}(x, y) \right| \rightarrow 0, \text{ as } n, m \rightarrow \infty.$$

From the condition of the Theorem 1 we can write

$$(27) \quad \left| L_{mn}^{(14)}(x, y) \right| \\ \leq c(\alpha, \beta) nm \sum_{i=\tau(n, m)}^{(m-1)/2} \frac{1}{i^{1-\alpha}} \sum_{j=\tau(n, m)}^{(n-1)/2} \frac{1}{j^{1-\beta}} \\ \left( \int_0^{\pi/m} \int_0^{\pi/n} |\Delta_{ij}^{mn} f(x+s, y+t)| ds dt \right) \\ \leq c(\alpha, \beta) nm \left\{ \sum_{i=\tau(n, m)}^{(m-1)/2} \frac{1}{i^{1-\alpha}} \sum_{j=i}^{(n-1)/2} \frac{\lambda_j}{j^{1-\beta}} \frac{1}{\lambda_j} \right. \\ \left. + \sum_{j=\tau(n, m)}^{(n-1)/2} \frac{1}{j^{1-\beta}} \sum_{i=j}^{(m-1)/2} \frac{1}{\lambda_i} \frac{\lambda_i}{i^{1-\alpha}} \right\} \\ \left( \int_0^{\pi/m} \int_0^{\pi/n} |\Delta_{ij}^{mn} f(x+s, y+t)| ds dt \right) \\ \leq c(\alpha, \beta) n \sum_{i=\tau(n, m)}^{(m-1)/2} \frac{\lambda_i}{i^{2-(\alpha+\beta)}} \left( \int_0^{\pi/n} \sup_x \sum_{j=i}^{(n-1)/2} \frac{1}{\lambda_j} |{}_2\Delta_j^n f(x, y+t)| dt \right)$$

$$\begin{aligned}
& +c(\alpha, \beta) m \sum_{j=\tau(n,m)}^{(n-1)/2} \frac{\lambda_j}{j^{2-(\alpha+\beta)}} \left( \int_0^{\pi/m} \sup_y \sum_{i=j}^{(m-1)/2} \frac{1}{\lambda_j} |\Delta_i^m f(x+s, y)| ds \right) \\
& \leq c(\alpha, \beta) (V_1 \Lambda(f) + V_2 \Lambda(f)) \sum_{j=\tau(n,m)}^{\infty} \frac{\lambda_j}{j^{2-(\alpha+\beta)}} \rightarrow 0,
\end{aligned}$$

as  $n, m \rightarrow \infty$ .

Combining (23)-(27) we conclude that

$$(28) \quad L_{mn}^{(1)}(x, y) \rightarrow 0 \quad \text{as } m, n \rightarrow \infty.$$

From (18), (19), (21), (22) and (28) we obtain

$$(29) \quad L_{mn}(x, y) \rightarrow 0 \quad \text{as } m, n \rightarrow \infty.$$

Finally, combining (7), (8), (16), (17) and (29) we get

$$I_{mn}^{(1)}(x, y) \rightarrow 0 \quad \text{as } m, n \rightarrow \infty.$$

Analogously, we can prove that

$$I_{mn}^{(k)}(x, y) \rightarrow 0 \quad \text{as } m, n \rightarrow \infty, \quad k = 2, 3, 4.$$

To complete the proof of Theorem 1, note that if  $f$  is continuous on some compact  $K$ , then the relations

$$\lim_{s, t \rightarrow 0} \varphi_i(x, y, s, t) = 0, \quad i = 1, 2, 3, 4,$$

hold uniformly on  $(x, y) \in K$  and all estimates in the proof also hold uniformly on  $(x, y) \in K$ . Hence the  $(C; -\alpha, \beta)$  means  $\sigma_{n,m}^{\alpha,\beta}(f; x, y)$  will converge to  $f$  uniformly on  $K$ .  $\square$

*Proof of Theorem 2.* It is not hard to see, that for any sequence  $\Lambda = \{\lambda_n\}$  satisfying (1) the class  $C(T^2) \cap P\Lambda BV$  is a Banach space with the norm

$$\|f\|_{P\Lambda BV} := \|f\|_C + P\Lambda V(f).$$

Denote

$$A_{i,j} := \left[ \frac{\pi i - \alpha\pi/2}{N + 1/2 - \alpha/2}, \frac{\pi(i+1) - \alpha\pi/2}{N + 1/2 - \alpha/2} \right) \times \left[ \frac{\pi j - \beta\pi/2}{N + 1/2 - \beta/2}, \frac{\pi(j+1) - \beta\pi/2}{N + 1/2 - \beta/2} \right)$$

and

$$W := \left\{ (i, j) : j < i < 2j, 1 < j < \frac{N-1}{2} \right\}.$$

Let

$$\begin{aligned}
f_N(x, y) & : = \sum_{(i,j) \in W} t_j \mathbf{1}_{A_{i,j}}(x, y) \sin[(N + 1/2 - \alpha/2)x + \alpha\pi/2] \\
& \quad \times \sin[(N + 1/2 - \beta/2)y + \beta\pi/2],
\end{aligned}$$

where

$$t_j := \left( \sum_{i=1}^j \frac{1}{\lambda_i} \right)^{-1}.$$

First, we prove that  $f \in P\Lambda BV$ . Indeed, let

$$y \in \left[ \frac{\pi j - \beta\pi/2}{N + 1/2 - \beta/2}, \frac{\pi(j+1) - \beta\pi/2}{N + 1/2 - \beta/2} \right).$$

Then it is evident that

$$\sum_i \frac{|f(I_i, y)|}{\lambda_i} \leq c \left( \sum_{i=j}^{2j-1} \frac{1}{\lambda_{2j-i}} \right) t_j \leq c < \infty,$$

consequently,

$$(30) \quad V_1 \Lambda(f) < \infty.$$

Let

$$x \in \left[ \frac{\pi i - \alpha\pi/2}{N + 1/2 - \alpha/2}, \frac{\pi(i+1) - \alpha\pi/2}{N + 1/2 - \alpha/2} \right)$$

then from construction of the function  $f$  we have

$$\sum_j \frac{|f(x, J_j)|}{\lambda_j} \leq c \sum_{j=[i/2]}^i \frac{t_j}{\lambda_{j-[i/2]+1}} \leq ct_{[i/2]} \left( \sum_{j=1}^{i-[i/2]+1} \frac{1}{\lambda_j} \right) \leq c < \infty.$$

Hence

$$(31) \quad V_2 \Lambda(f) < \infty.$$

Combining (30) and (31) and we conclude that  $f \in P\Lambda BV$ .

From (2)-(5) we can write

$$\begin{aligned} (32) \quad & \pi^2 \sigma_{N,N}^{(-\alpha, -\beta)} f_N(0, 0) \\ &= \int_{T^2} f_N(x, y) K_N^{-\alpha}(x) K_N^{-\beta}(y) dx dy \\ &= \sum_{(i,j) \in W} t_j \int_{A_{i,j}} \sin[(N + 1/2 - \alpha/2)x + \alpha\pi/2] \sin[(N + 1/2 - \beta/2)y + \beta\pi/2] \\ & \quad \times O\left(\frac{1}{Nx^2}\right) O\left(\frac{1}{Ny^2}\right) dx dy \\ &+ \sum_{(i,j) \in W} t_j \int_{A_{i,j}} \sin[(N + 1/2 - \alpha/2)x + \alpha\pi/2] \frac{\sin^2[(N + 1/2 - \beta/2)y + \beta\pi/2]}{A_N^{-\beta} (2 \sin y/2)^{1-\beta}} \\ & \quad \times O\left(\frac{1}{Nx^2}\right) dx dy \\ &+ \sum_{(i,j) \in W} t_j \int_{A_{i,j}} \frac{\sin^2[(N + 1/2 - \alpha/2)x + \alpha\pi/2]}{A_N^{-\alpha} (2 \sin x/2)^{1-\alpha}} \sin[(N + 1/2 - \beta/2)y + \beta\pi/2] \\ & \quad \times O\left(\frac{1}{Ny^2}\right) dx dy \end{aligned}$$

$$\begin{aligned}
& + \sum_{(i,j) \in W} t_j \int_{A_{i,j}} \frac{\sin^2 [(N+1/2 - \alpha/2)x + \alpha\pi/2]}{A_N^{-\alpha} (2 \sin x/2)^{1-\alpha}} \frac{\sin^2 [(N+1/2 - \beta/2)y + \beta\pi/2]}{A_N^{-\beta} (2 \sin y/2)^{1-\beta}} dx dy \\
& =: \sum_{k=1}^4 F_N^{(k)}(x, y)
\end{aligned}$$

It is easy to show that

$$\begin{aligned}
(33) \quad & \left| F_N^{(1)}(x, y) \right| \leq c \sum_{(i,j) \in W} \frac{t_j}{ij} \\
& = c \sum_{j=1}^{[(N-1)/2]} \frac{t_j}{j} \sum_{i=j+1}^{2j-1} \frac{1}{i} \leq c \sum_{j=1}^{[(N-1)/2]} \frac{t_j}{j},
\end{aligned}$$

$$\begin{aligned}
(34) \quad & \left| F_N^{(2)}(x, y) \right| \leq c(\alpha, \beta) \sum_{(i,j) \in W} \frac{t_j}{ij^{1-\beta}} \\
& = c(\alpha, \beta) \sum_{j=1}^{[(N-1)/2]} \frac{t_j}{j^{1-\beta}} \sum_{i=j+1}^{2j-1} \frac{1}{i} \\
& \leq c(\alpha, \beta) \sum_{j=1}^{[(N-1)/2]} \frac{t_j}{j^{1-\beta}},
\end{aligned}$$

$$\begin{aligned}
(35) \quad & \left| F_N^{(3)}(x, y) \right| \leq c(\alpha, \beta) \sum_{j=1}^{[(N-1)/2]} \frac{t_j}{j} \sum_{i=j+1}^{2j-1} \frac{1}{i^{1-\alpha}} \\
& \leq c(\alpha, \beta) \sum_{j=1}^{[(N-1)/2]} \frac{t_j}{j^{1-\alpha}}.
\end{aligned}$$

From the construction of the function  $f_N$  we can write

$$\begin{aligned}
(36) \quad & \left| F_N^{(4)}(x, y) \right| \\
& = \frac{1}{(N+1/2 - \alpha/2)(N+1/2 - \beta/2)} \sum_{(i,j) \in W} t_j \\
& \int_{\pi i}^{\pi(i+1)} \int_{\pi j}^{\pi(j+1)} \frac{\sin^2 u}{A_N^{-\alpha} \left( 2 \sin \frac{u - \alpha\pi/2}{2(N+1/2 - \alpha/2)} \right)^{1-\alpha}} \frac{\sin^2 v}{A_N^{-\beta} \left( 2 \sin \frac{v - \beta\pi/2}{2(N+1/2 - \beta/2)} \right)^{1-\beta}} du dv \\
& \geq \frac{c(\alpha, \beta) N^{\alpha+\beta}}{N^2} \sum_{(i,j) \in W} t_j \frac{N^{2-(\alpha+\beta)}}{i^{1-\alpha} j^{1-\beta}} \int_{\pi i}^{\pi(i+1)} \sin^2 u du \int_{\pi j}^{\pi(j+1)} \sin^2 v dv
\end{aligned}$$



$$\begin{aligned}
&\geq c(\alpha, \beta) \sum_{(i,j) \in W} \frac{t_j}{j^{1-\beta}} \frac{1}{i^{1-\alpha}} \geq c(\alpha, \beta) \sum_{j=1}^{[(N-1)/2]} \frac{t_j}{j^{1-\beta}} \sum_{i=j+1}^{2j-1} \frac{1}{i^{1-\alpha}} \\
&\geq c(\alpha, \beta) \sum_{j=1}^{[(N-1)/2]} \frac{t_j}{j^{1-(\beta+\alpha)}}.
\end{aligned}$$

Since  $\frac{1}{j^{1-\alpha}} + \frac{1}{j^{1-\beta}} = o\left(\frac{1}{j^{1-(\alpha+\beta)}}\right)$  as  $j \rightarrow \infty$ , from (32)-(36) we conclude that if  $j_0$  is big enough and  $N > 2j_0$ , then

$$(37) \quad \pi^2 \left| \sigma_{N,N}^{(-\alpha, -\beta)} f_N(0, 0) \right| \geq c(\alpha, \beta) \sum_{j=j_0}^{[(N-1)/2]} \frac{t_j}{j^{1-(\beta+\alpha)}}.$$

Let  $\lambda_j = j^{1-(\alpha+\beta)} \gamma_j$ ,  $\gamma_j \geq \gamma_{j+1}$ ,  $j = 1, 2, \dots$ . Then we can write

$$\frac{1}{t_j} = \sum_{i=1}^j \frac{1}{\lambda_i} = \sum_{i=1}^j \frac{1}{i^{1-(\alpha+\beta)} \gamma_i} \leq c(\alpha, \beta) \frac{j^{\alpha+\beta}}{\gamma_j}.$$

Consequently,

$$(38) \quad t_j j^{\alpha+\beta} \geq c(\alpha, \beta) \gamma_j.$$

Combining (37) and (38) we obtain

$$\begin{aligned}
(39) \quad &\pi^2 \left| \sigma_{N,N}^{(-\alpha, -\beta)} f_N(0, 0) \right| \geq c(\alpha, \beta) \sum_{j=j_0}^{[(N-1)/2]} \frac{\gamma_j}{j} \\
&= c(\alpha, \beta) \sum_{j=j_0}^{[(N-1)/2]} \frac{t_j}{j^{2-(\beta+\alpha)}} \rightarrow \infty \text{ as } N \rightarrow \infty.
\end{aligned}$$

Applying the Banach-Steinhaus Theorem, from (39) we obtain that there exists a continuous function  $f \in P\Lambda BV$  such that

$$\sup_N \left| \sigma_{N,N}^{(-\alpha, -\beta)} f_N(0, 0) \right| = +\infty.$$

□

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